CS 4495 Computer Vision

N-Views (1) – Homographies and Projection

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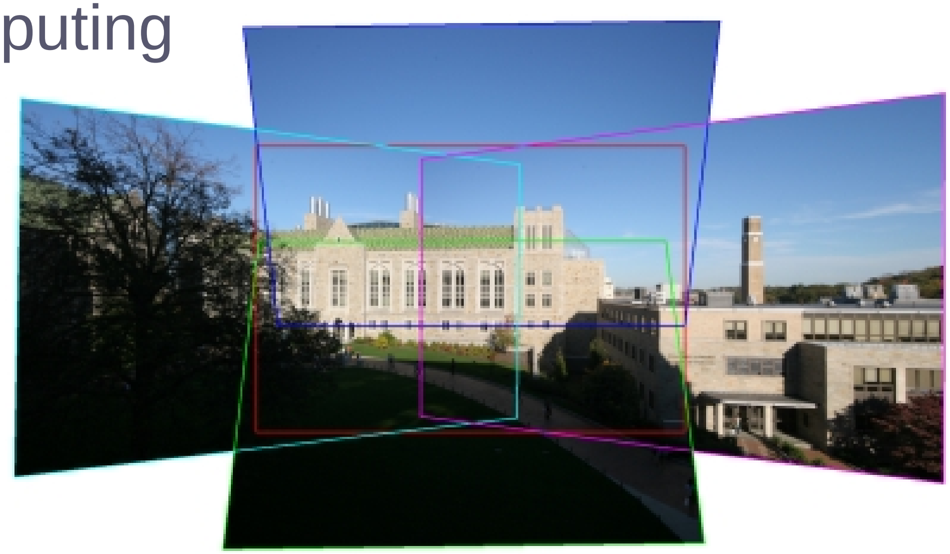


Administrivia

- PS 2:
  - Get SDD and Normalized Correlation working for a given windows size – say 5x5. Then try on a few window sizes.
  - If too slow, resize the images (imresize or take some of the middle) and get your code working. Then try on full images (which are reduced already!).
  - Results not perfect on even test images? Should they be?
  - Yes you can use normxcorr2 (it did say this!)
  - Some loops are OK. For SSD you might have 3 nested loops (row, col, disp) but shouldn't be looping over pixels.
  - **EXPERT**: There is a fast way: shift entire right image by d. Subtract from left, square, then box filter. This is *all* the numbers you need for disparity d for all pixel locations. Do this over all d and you have all the data you need to compute best match for every pixel. .

Administrivia

- Now: Multiple-views
  - FP 7.1, 8 (all)
- Today: First half of 2-Views …

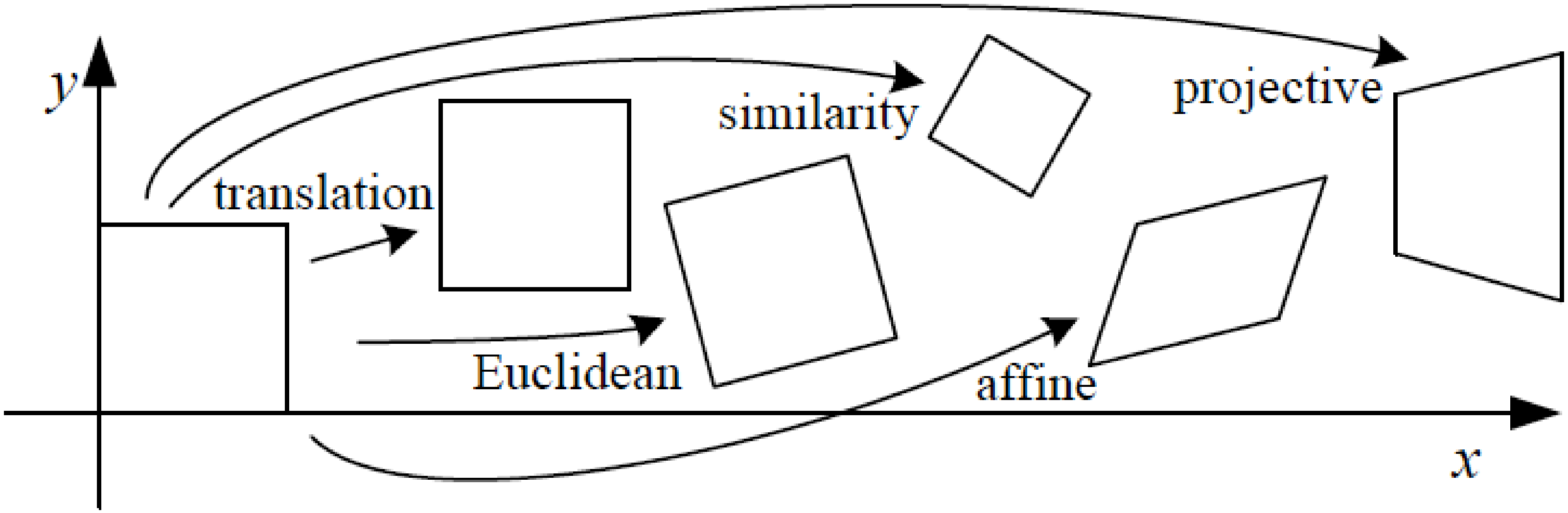
Two views…and two lectures

- Projective transforms from image to image

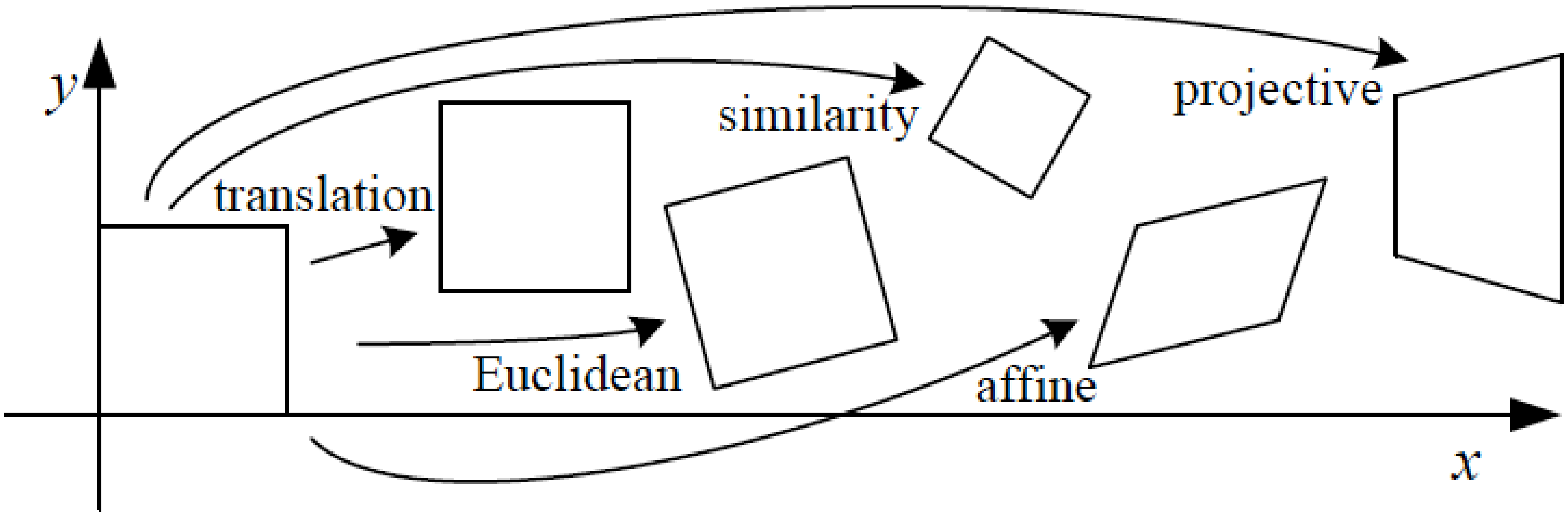
- Some more projective geometry
  - Points and lines and planes

- Two arbitrary views of the same scene
  - Calibrated – “Essential Matrix”
  - Two uncalibrated cameras “Fundamental Matrix”
    - Gives epipolar lines

2D Transformations



2D Transformations

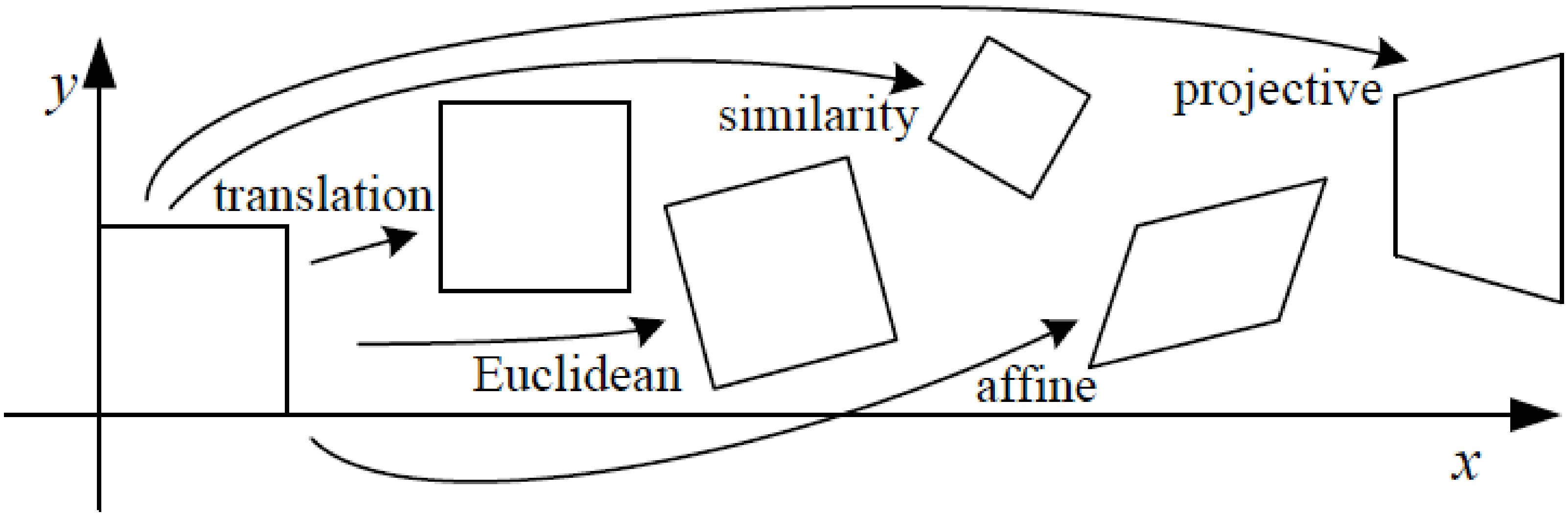


Example: translation

$$x' = x + t$$

$$\begin{bmatrix} \, \\ \, \end{bmatrix} = \begin{bmatrix} \, \\ \, \end{bmatrix} + \begin{bmatrix} tx \\ ty \end{bmatrix}$$

2D Transformations

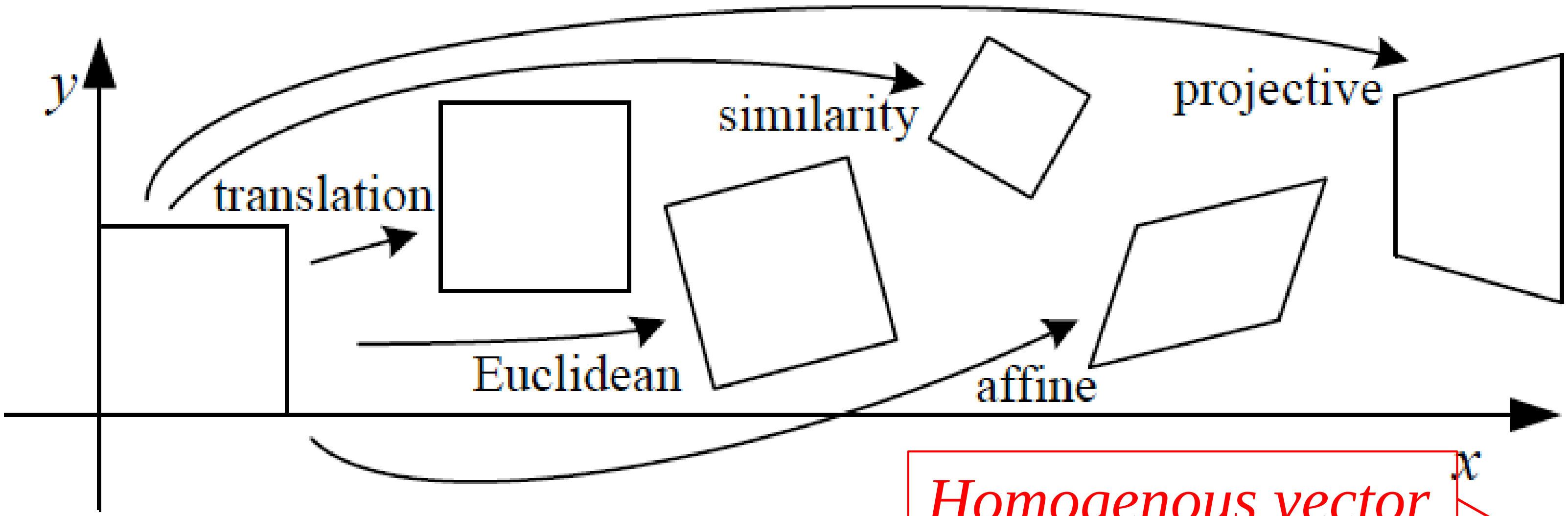


Example: translation

$$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{t} \qquad \boldsymbol{x}' = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{t} \end{bmatrix} \bar{\boldsymbol{x}}$$

$$\begin{bmatrix} \; \\ \; \end{bmatrix} = \begin{bmatrix} \; \\ \; \end{bmatrix} + \begin{bmatrix} tx \\ ty \end{bmatrix} \qquad \begin{bmatrix} \; \\ \; \end{bmatrix} = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \end{bmatrix} \cdot \begin{bmatrix} \; \\ \; \\ 1 \end{bmatrix}$$

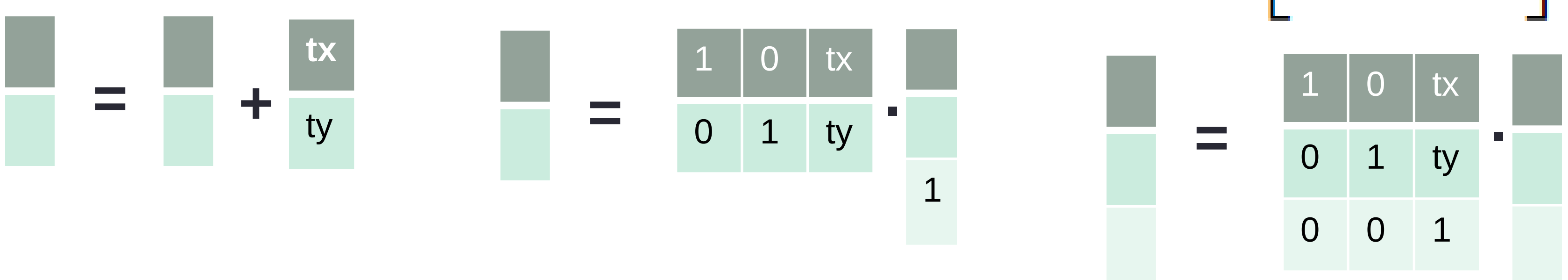
2D Transformations



Homogenous vector

Example: translation

$$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{t} \qquad \boldsymbol{x}' = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{t} \end{bmatrix} \bar{\boldsymbol{x}} \qquad \bar{\boldsymbol{x}}' = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix} \bar{\boldsymbol{x}}$$



[BTW: Now we can chain transformations]

Projective Transformations

- *Projective* transformations: for 2D images it's a 3x3 matrix applied to homogenous coordinates

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Special Projective Transformations

- **Translation**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

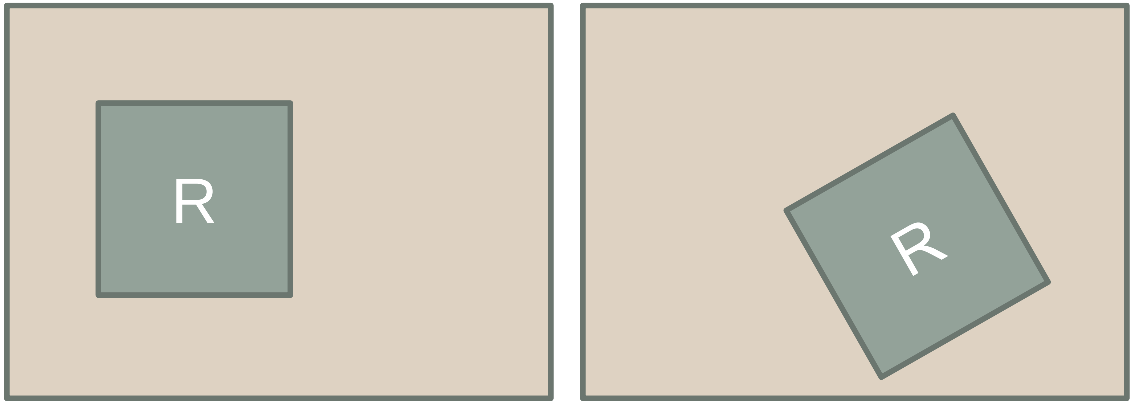
- **Preserves:**
  - Lengths/Areas
  - Angles
  - Orientation
  - Lines

Special Projective Transformations

- **Euclidean (Rigid body)**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & t_x \\ \sin(\theta) & \cos(\theta) & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- **Preserves:**
  - Lengths/Areas
  - Angles
  - Lines

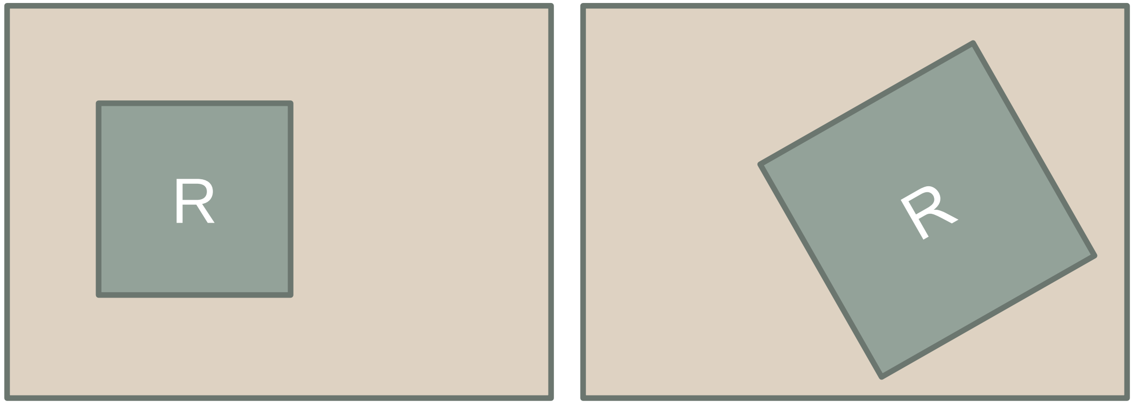


Special Projective Transformations

- **Similarity (trans, rot, scale) transform**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a\cos(\theta) & -a\sin(\theta) & t_x \\ a\sin(\theta) & a\cos(\theta) & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- **Preserves:**
  - Ratios of Areas
  - Angles
  - Lines

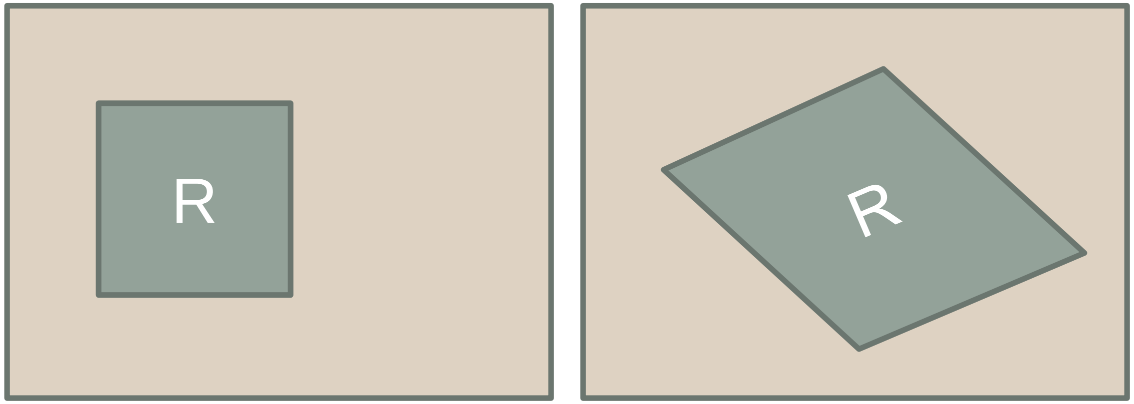


Special Projective Transformations

- **Affine transform**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- **Preserves:**
  - Parallel lines
  - Ratio of Areas
  - Lines



Projective Transformations

- Remember, these are homogeneous coordinates

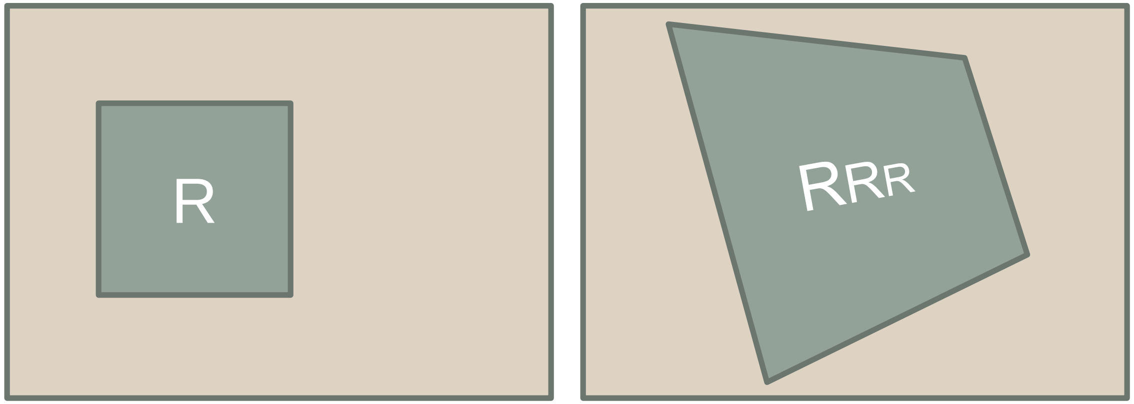
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \cong \begin{bmatrix} sx' \\ sy' \\ s \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Projective Transformations

- General **projective transform** (or *Homography)*

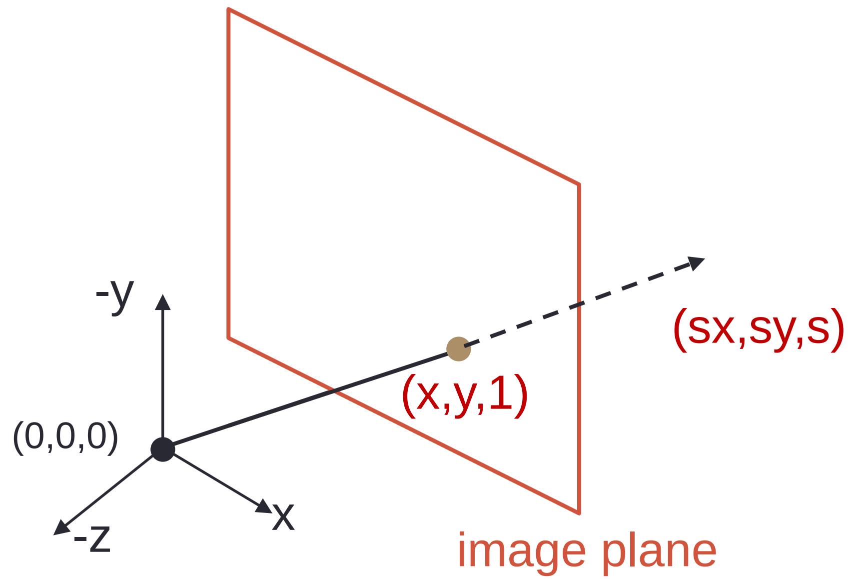
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \cong \begin{bmatrix} sx' \\ sy' \\ s \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- **Preserves:**
  - Lines
  - Also cross ratios (maybe later)



The projective plane

- What is the geometric intuition of using homogenous coordinates ?
  - a point in the image is a *ray* in projective space

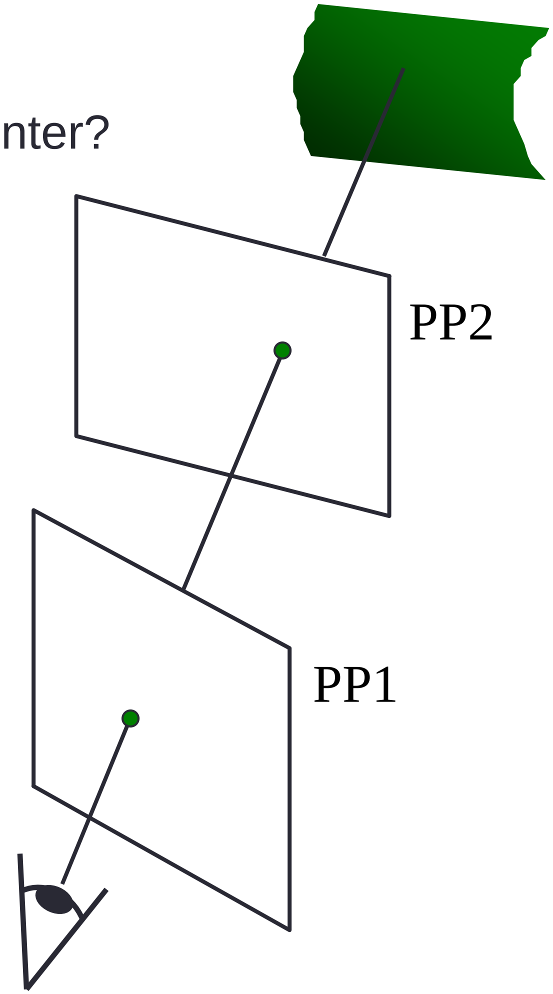


- Each *point* (x,y) on the plane (at z=1) is represented by a *ray* (sx,sy,s)
  – all points on the ray are equivalent: $(x, y, 1) \cong (sx, sy, s)$

Image reprojection

The irrelevant world!

- Basic question
  - How to relate two images from the same camera center?
    - how to map a pixel from projective plane PP1 to PP2



Answer

- Cast a ray through each pixel in PP1
- Draw the pixel where that ray intersects PP2

Observation:
Rather than thinking of this as a 3D reprojection, think of it as a 2D **image warp** from one image to another.

Source: Alyosha Efros

Application: Simple mosaics

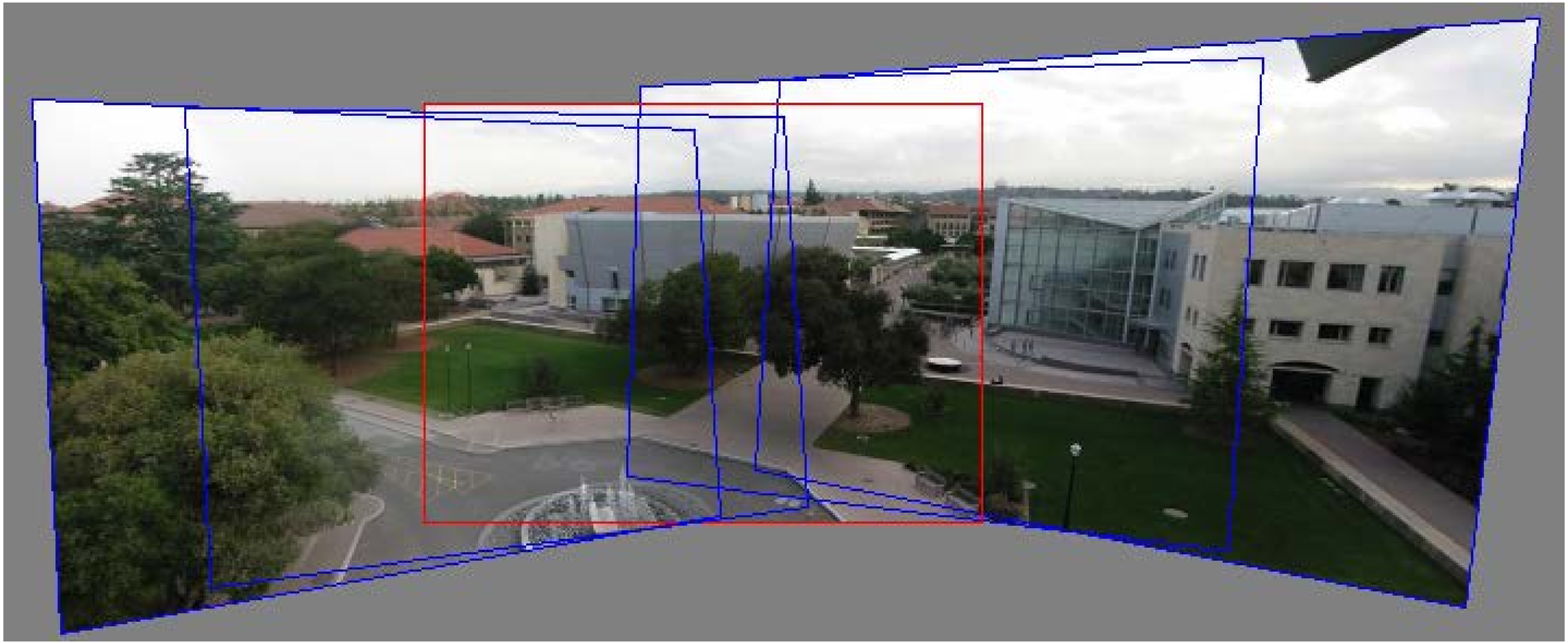


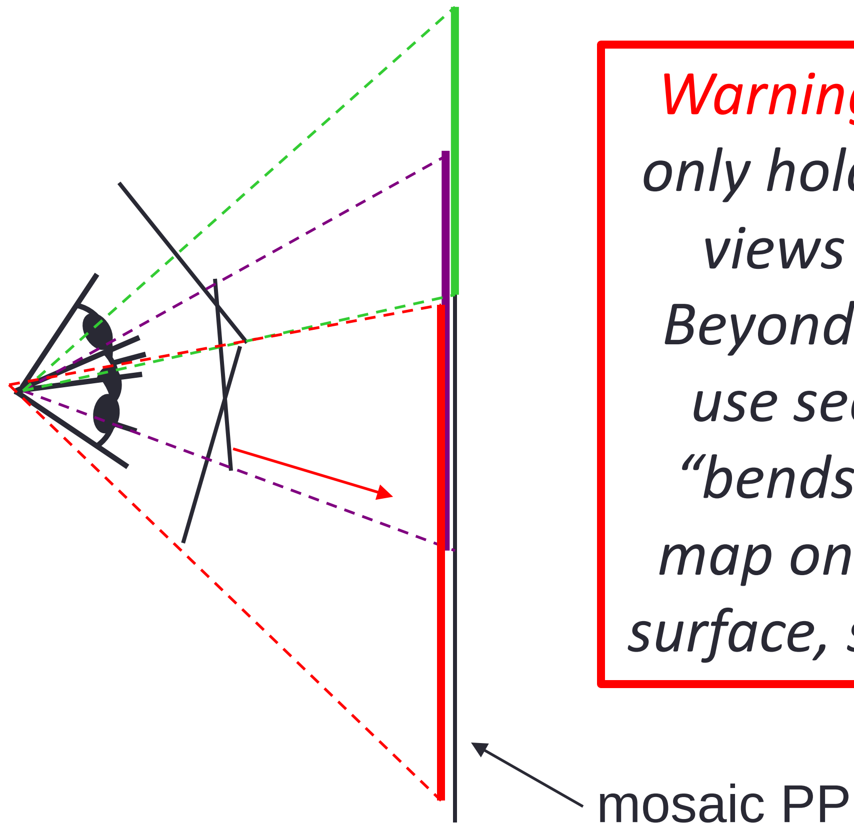
Image from http://graphics.cs.cmu.edu/courses/15-463/2010_fall/

How to stitch together a panorama (a.k.a. mosaic)?

- Basic Procedure
  - Take a sequence of images from the same position
    - Rotate the camera about its optical center
  - Compute transformation between second image and first
  - Transform the second image to overlap with the first
  - Blend the two together to create a mosaic
  - (If there are more images, repeat)
- …but wait, why should this work at all?
  - What about the 3D geometry of the scene?
  - Why aren’t we using it?

Source: Steve Seitz

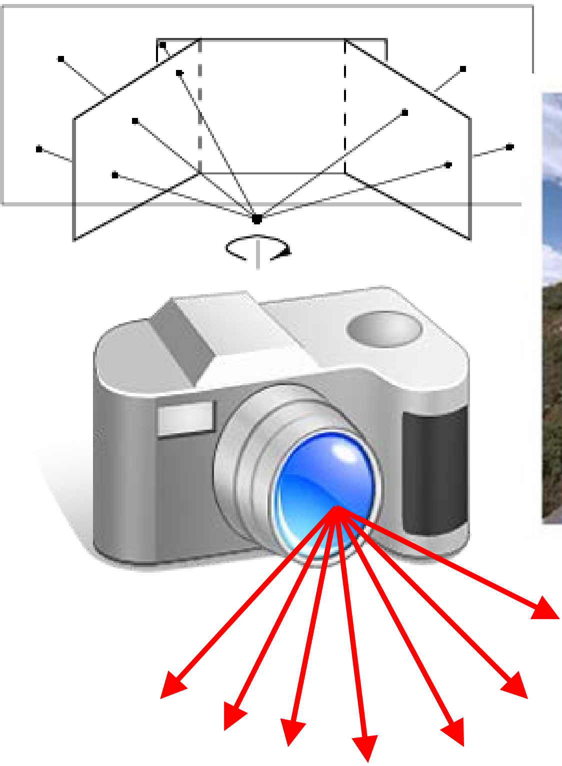
Image reprojection



Warning: This model only holds for angular views up to 180°. Beyond that need to use sequence that "bends the rays" or map onto a different surface, say, a cylinder.

- The mosaic has a natural interpretation in 3D
  - The images are reprojected onto a common plane
  - The mosaic is formed on this plane

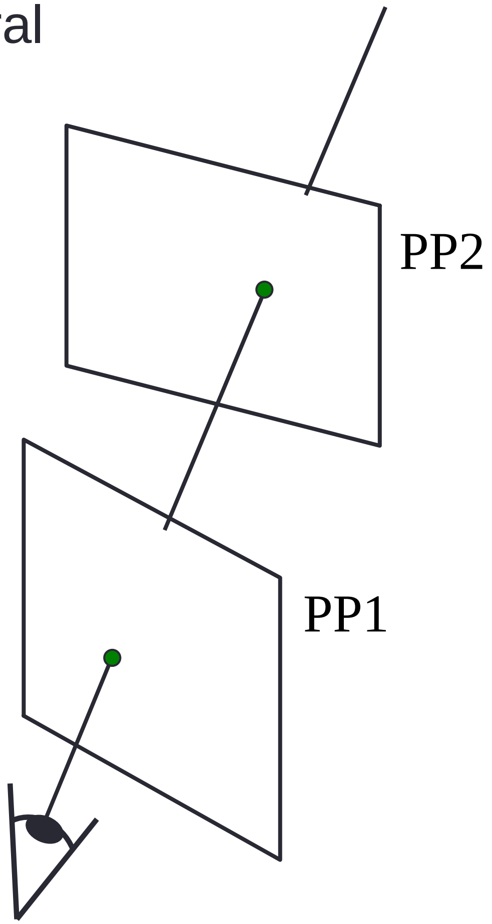
Mosaics



Obtain a wider angle view by combining multiple images ***all of which are taken from the same camera center***.

Image reprojection: Homography

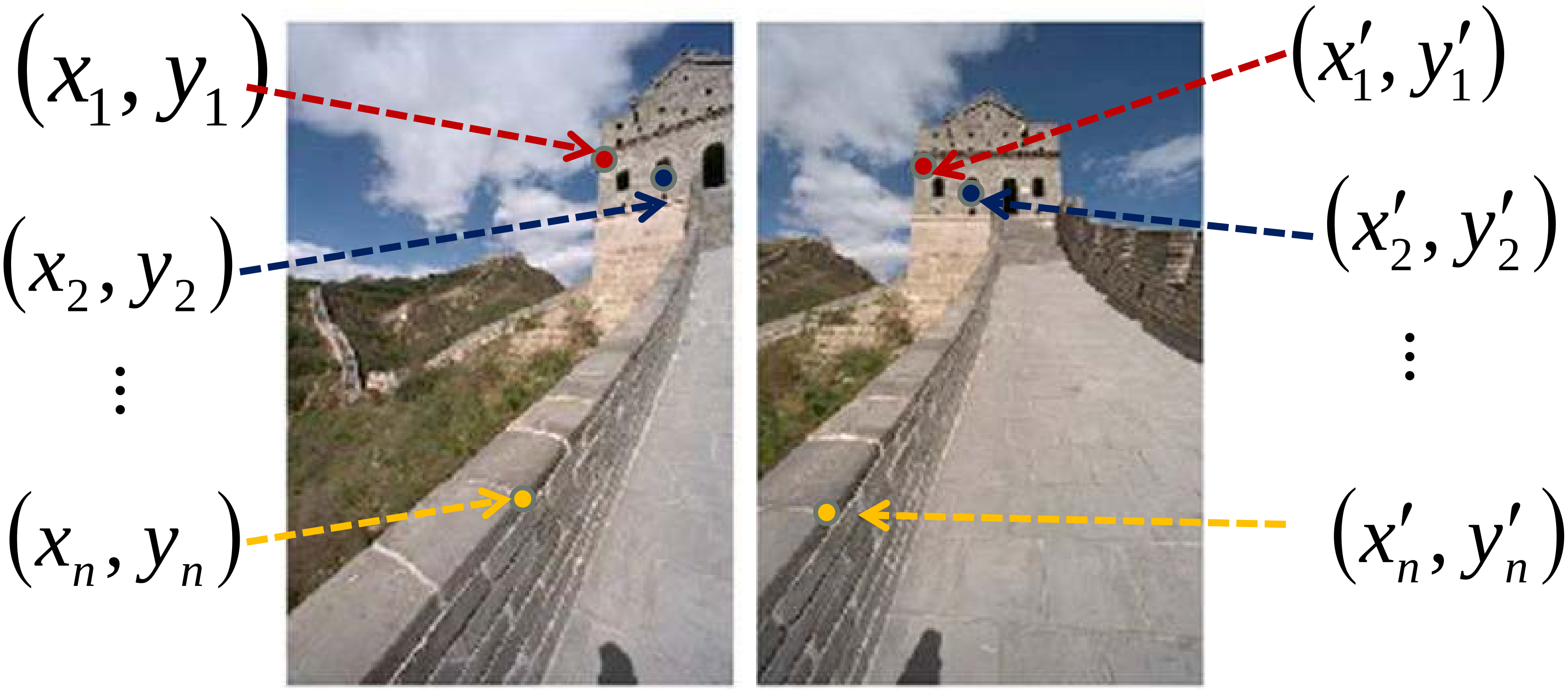
- A projective transform is a mapping between any two PPs with the same center of projection
  - rectangle should map to arbitrary quadrilateral
  - parallel lines aren't
  - but must preserve straight lines
- called **Homography**



$$\underset{\mathbf{p'}}{\begin{bmatrix} wx' \\ wy' \\ w \end{bmatrix}} = \underset{\mathbf{H}}{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}} \underset{\mathbf{p}}{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}$$

Source: Alyosha Efros

Homography



To **compute** the homography given pairs of corresponding points in the images, we need to set up an equation where the parameters of **H** are the unknowns…

Solving for homographies

$$\mathbf{p'} = \mathbf{Hp}$$

$$\begin{bmatrix} wx' \\ wy' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

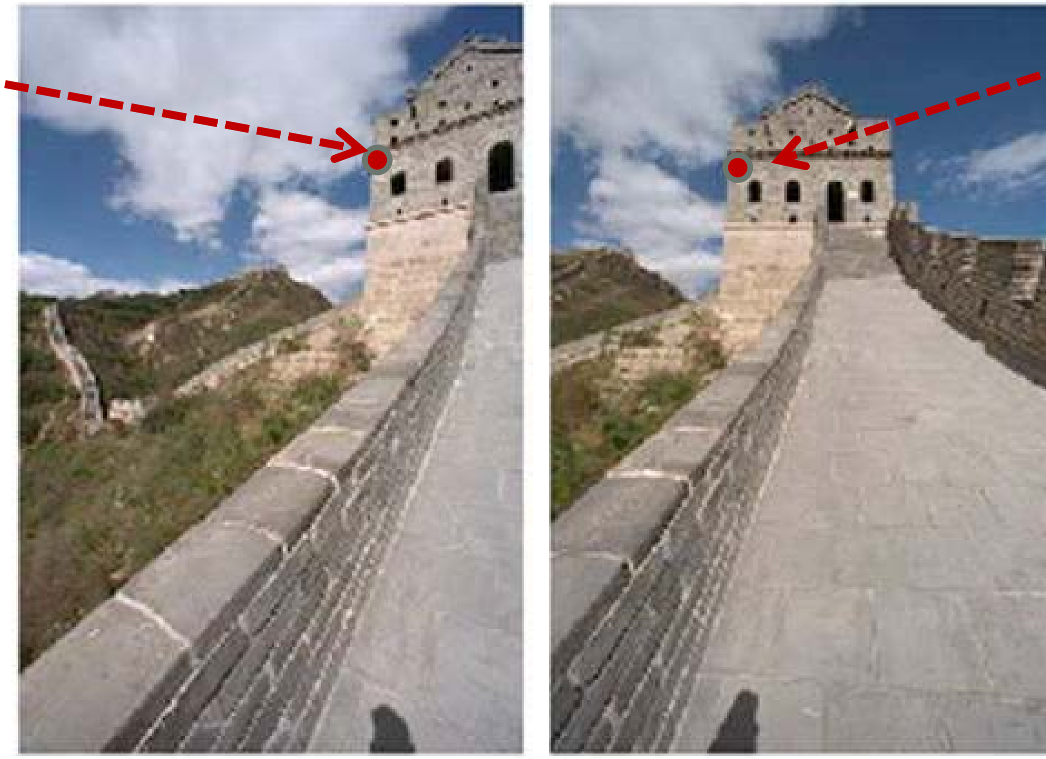
- Can set scale factor i=1. So, there are 8 unknowns.
- Set up a system of linear equations $\mathbf{Ah} = \mathbf{b}$
- where vector of unknowns h = [a,b,c,d,e,f,g,h]T
- Need at least 4 points for 8 eqs, but the more the better…
- Solve for h. If overconstrained, solve using least-squares:

$$\min \left\| \mathbf{Ah} - \mathbf{b} \right\|^2$$

- Look familiar? (If don't set ***i*** to 1 can use SVD)
- >> help mldivide

Homography

$$(x, y)$$



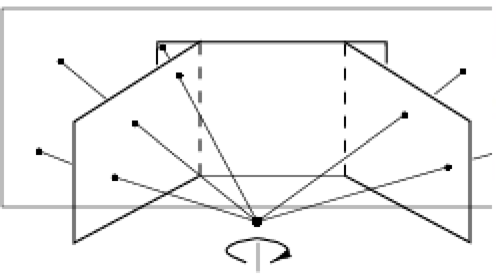
$$\left({}^{wx'}\!/_{w}, {}^{wy'}\!/_{w} \right) = (x', y')$$

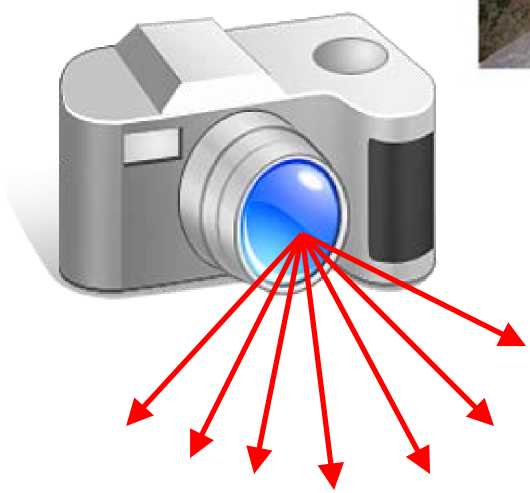
To **apply** a given homography **H**

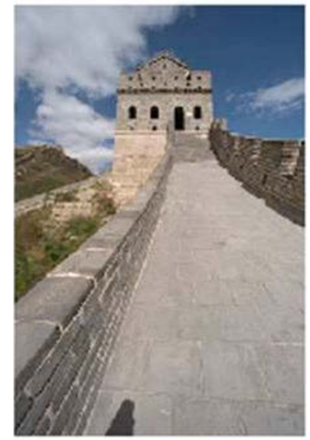
- Compute **p’** = **Hp** (regular matrix multiply)
- Convert **p’** from homogeneous to image coordinates

$$\underset{\mathbf{p'}}{\begin{bmatrix} wx' \\ wy' \\ w \end{bmatrix}} = \underset{\mathbf{H}}{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}} \underset{\mathbf{p}}{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}$$

Mosaics







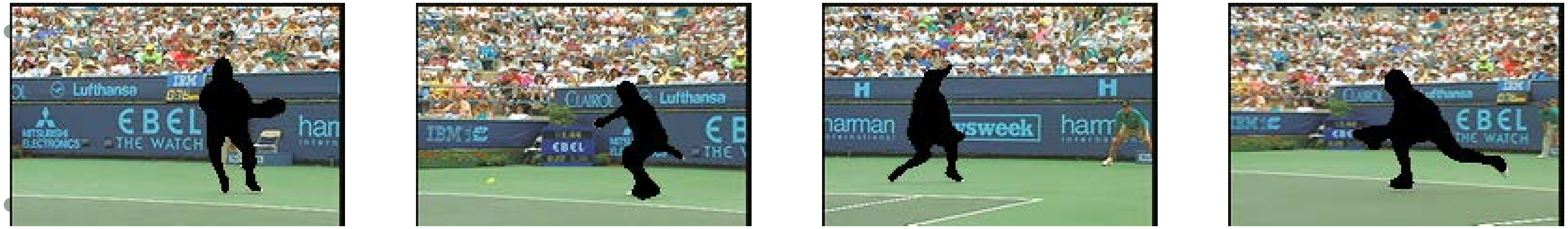
. . .

image from S. Seitz

Combine images with the computed homographies…

Mosaics for Video Coding

- Convert masked images into a background sprite for content-based coding



=

Homographies and 3D planes

- Remember this:

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \cong \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

- Suppose the 3D points are on a plane:

$$aX + bY + cZ + d = 0$$

Homographies and 3D planes

- On the plane *[a b c d]* can replace Z:

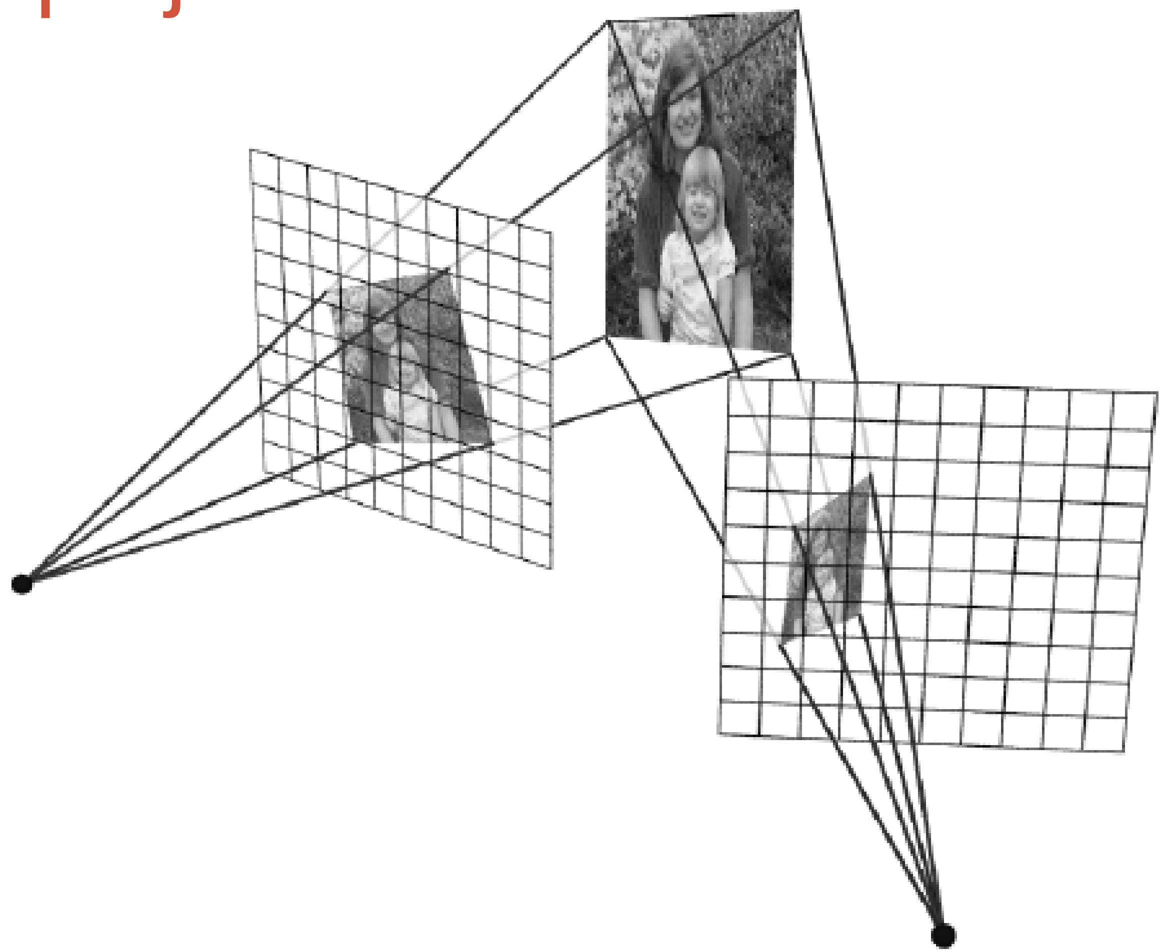
$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \cong \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \end{bmatrix} \begin{bmatrix} X \\ Y \\ (aX + bY + d)/(-c) \\ 1 \end{bmatrix}$$

- So, can put the Z coefficients into the others:

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \cong \begin{bmatrix} m'_{00} & m'_{01} & 0 & m'_{03} \\ m'_{10} & m'_{11} & 0 & m'_{13} \\ m'_{20} & m'_{21} & 0 & m'_{23} \end{bmatrix} \begin{bmatrix} X \\ Y \\ (aX + bY + d)/(-c) \\ 1 \end{bmatrix}$$

3x3 Homography!

Image reprojection



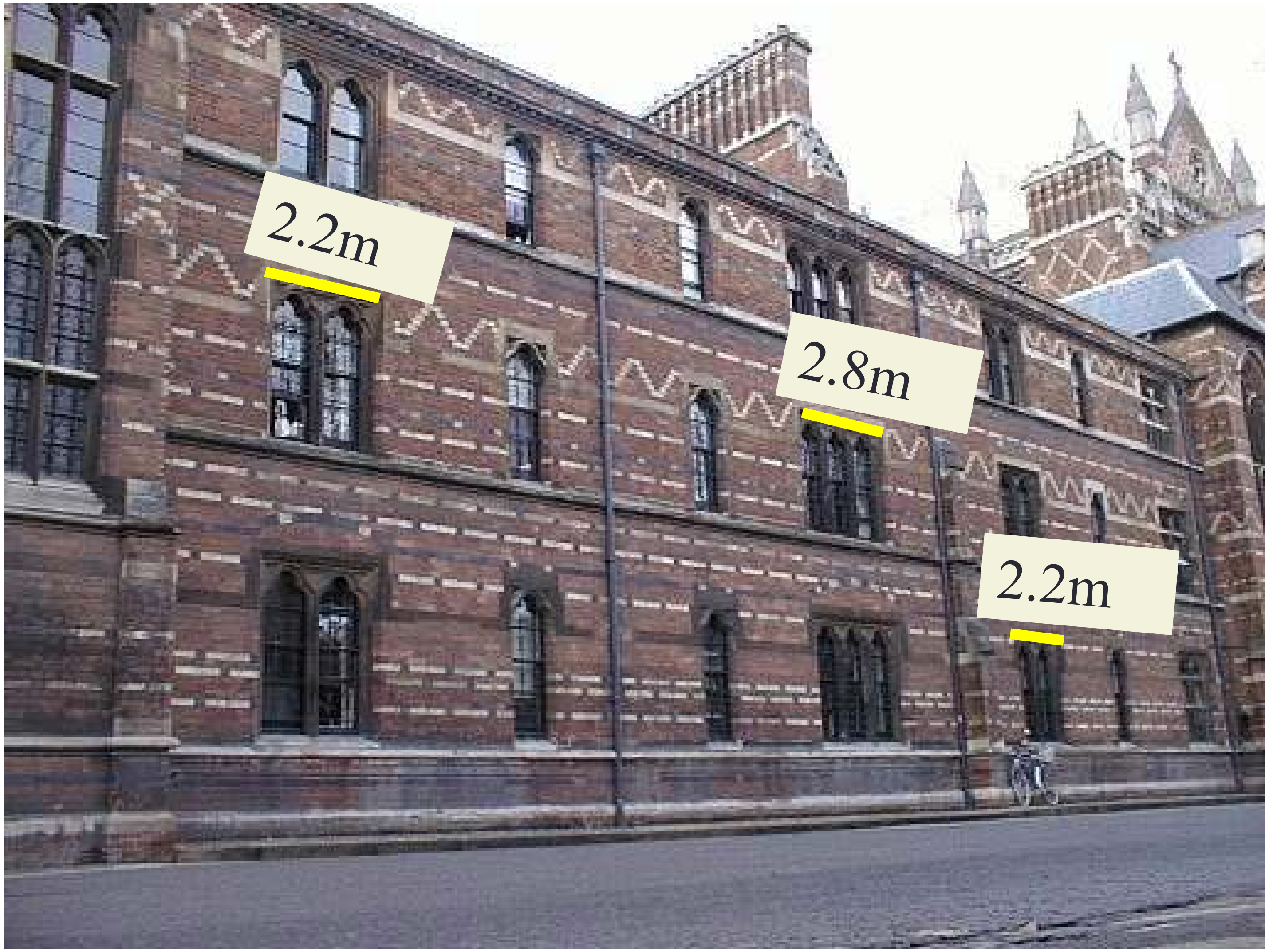
- Mapping between planes is a homography. Whether a plane in the world to the image or between image planes.

What else: Rectifying Slanted Views of Planes

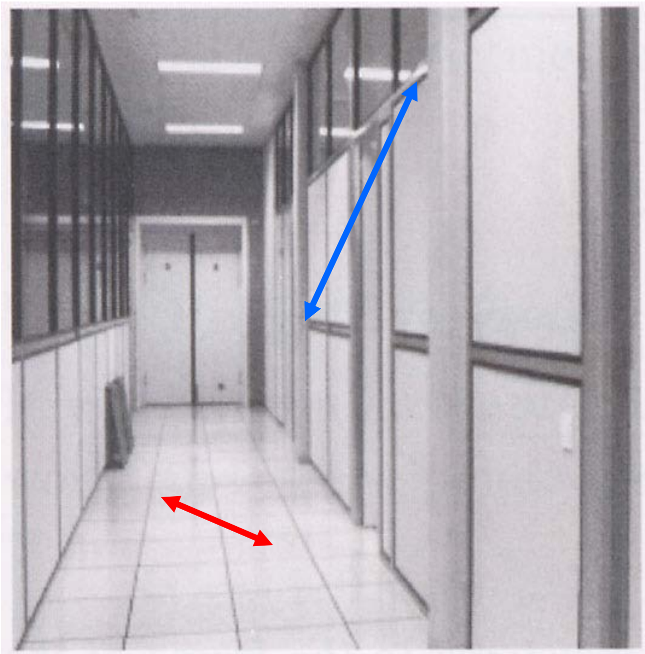
Rectifying slanted views

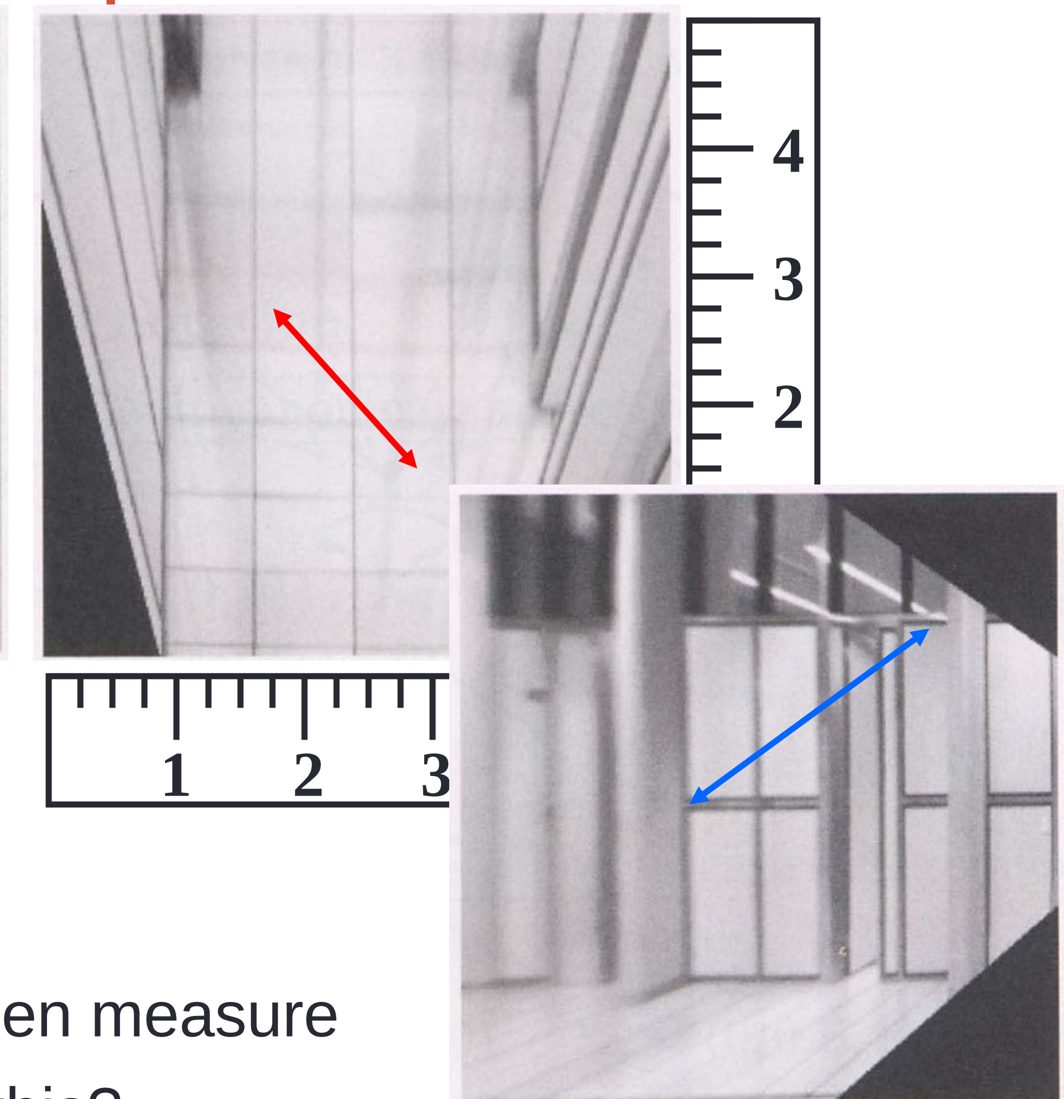
Corrected image (front-to-parallel)

Measuring distances



Measurements on planes

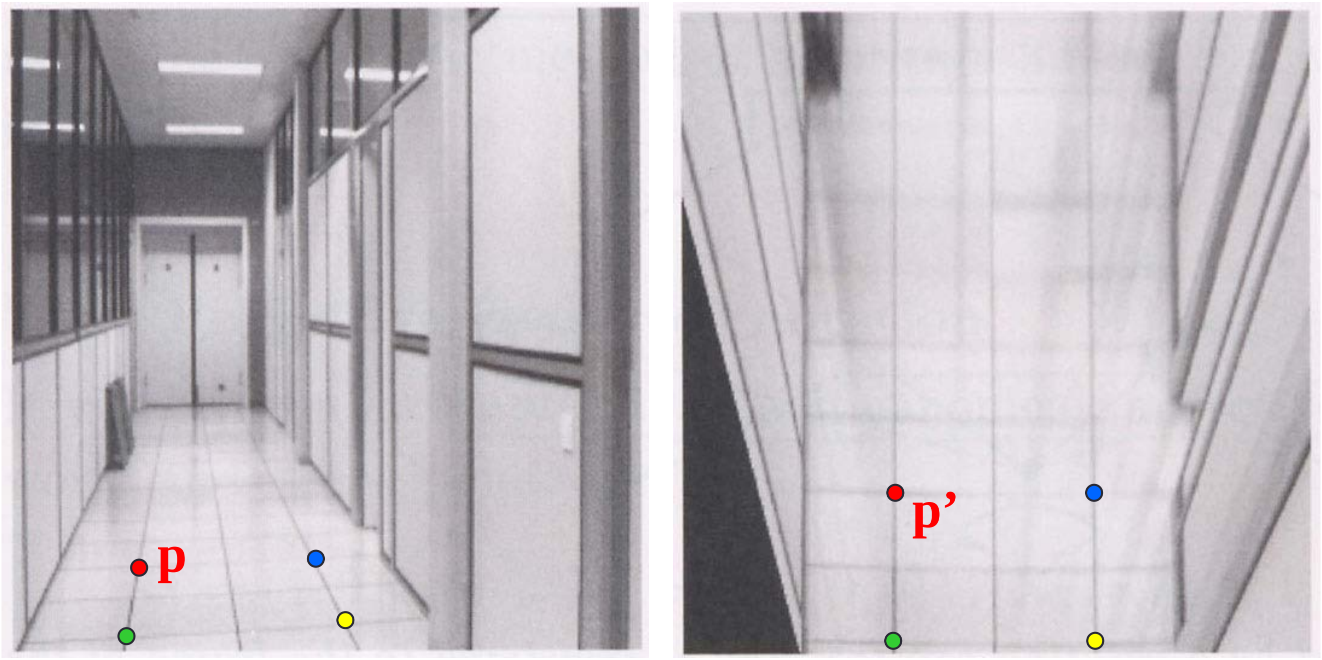




Approach: unwarp then measure

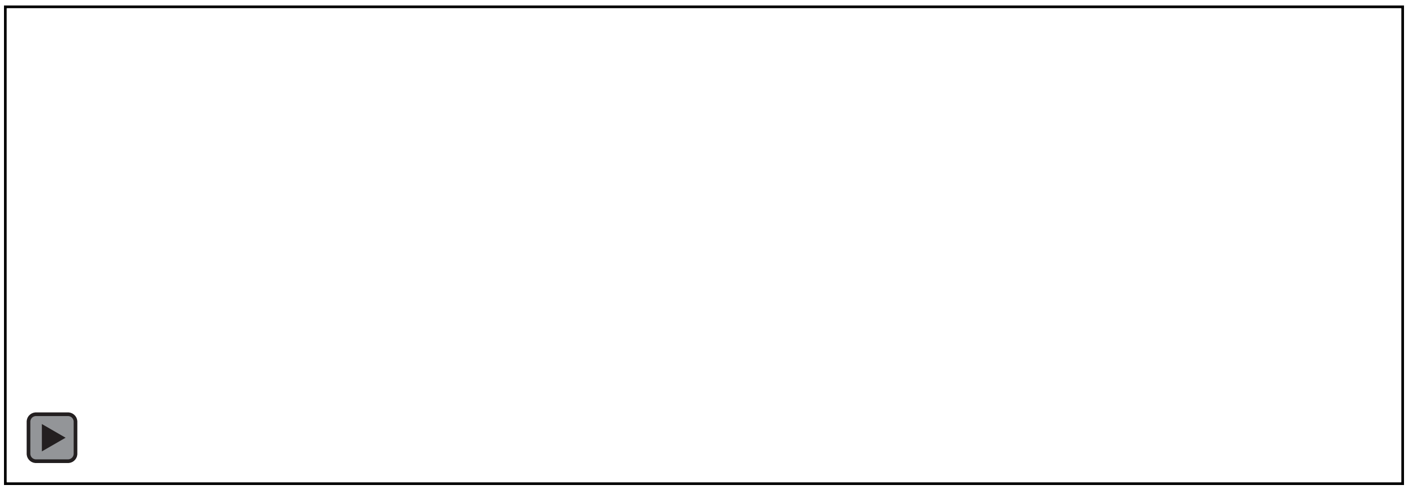
What kind of warp is this?
Homography…

Image rectification



A planar rectangular grid in the scene. Map it into a rectangular grid in the image.

Some other images of rectangular grids…



Who needs a blimp?

Same pixels – via a homography

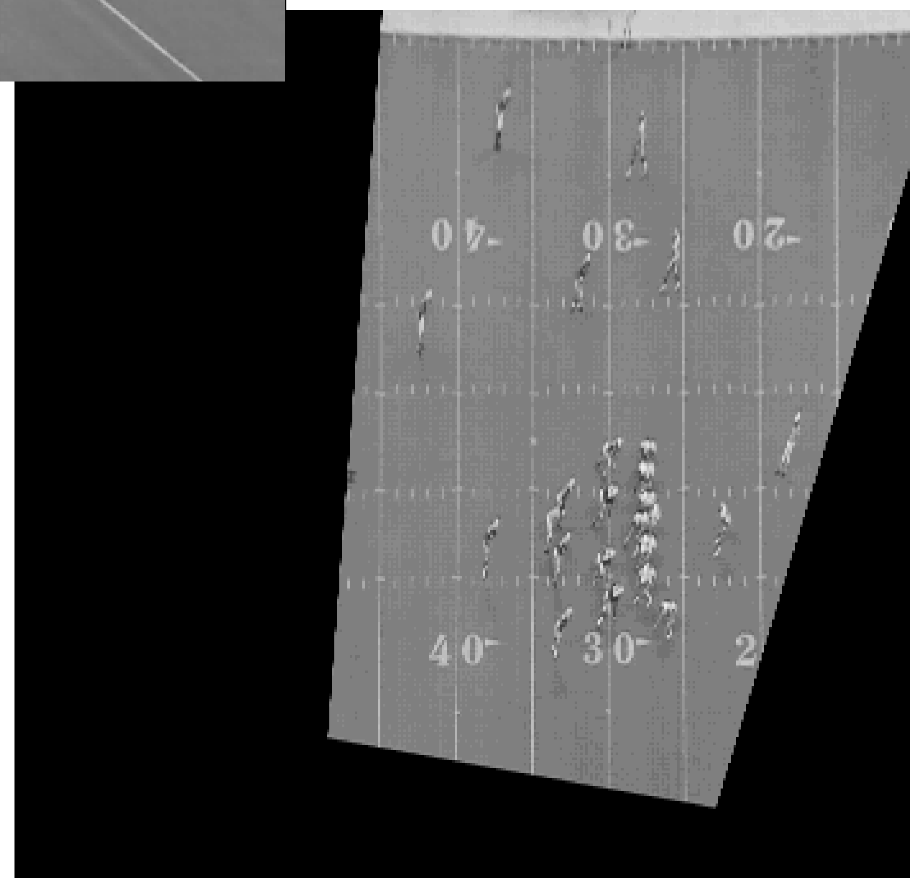
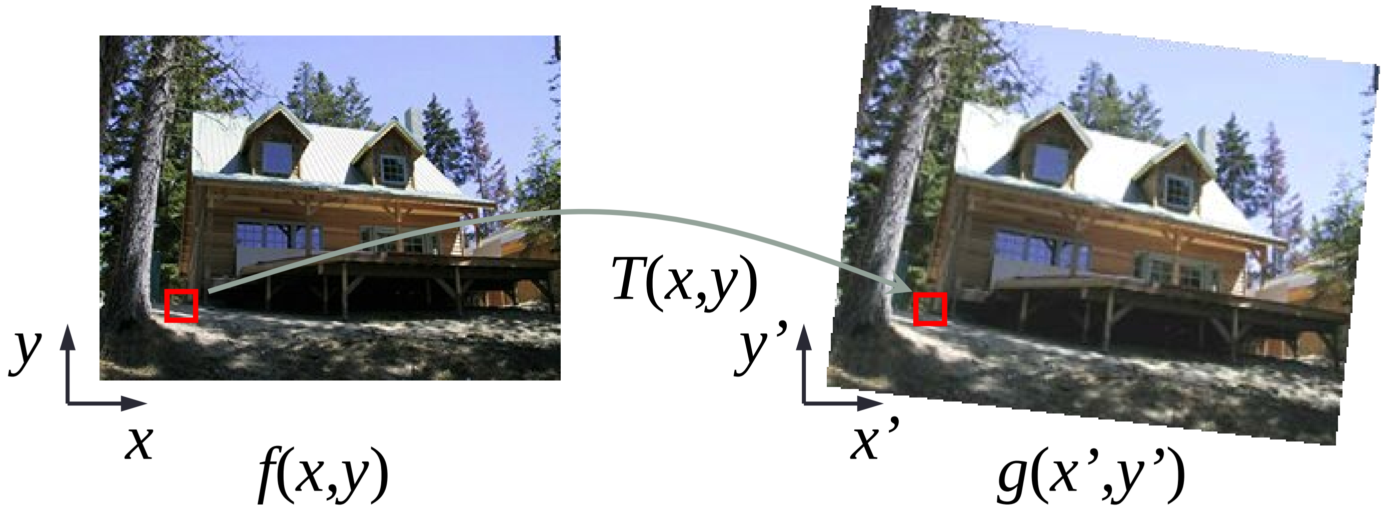


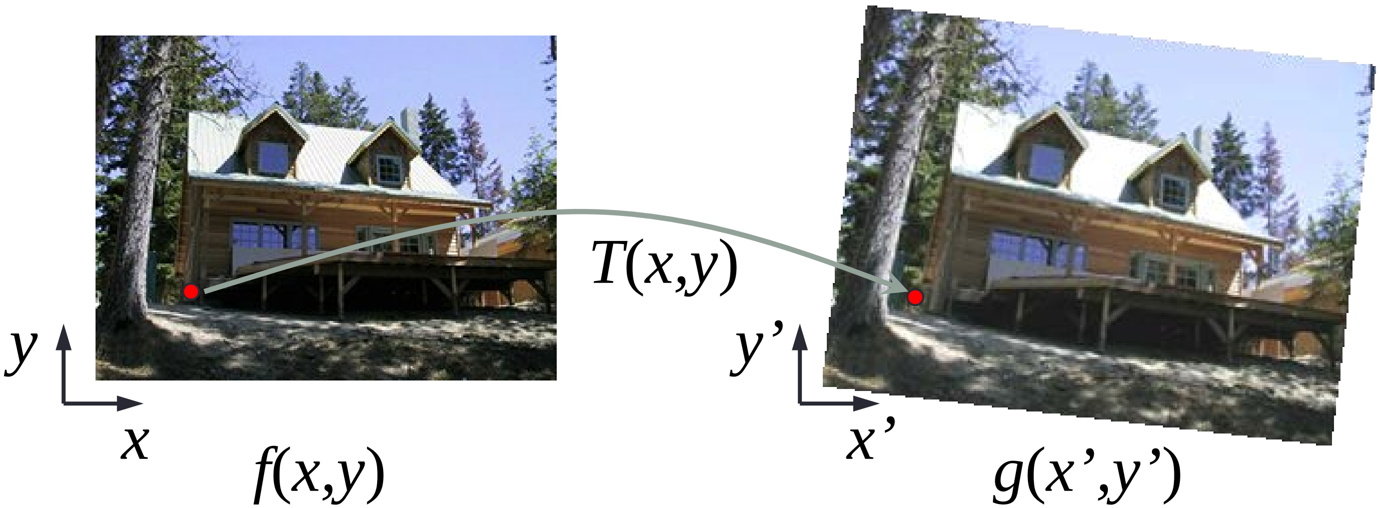
Image warping



Given a coordinate transform and a source image $f(x,y)$, how do we compute a transformed image $g(x',y') = f(T(x,y))$?

Slide from Alyosha Efros, CMU

Forward warping

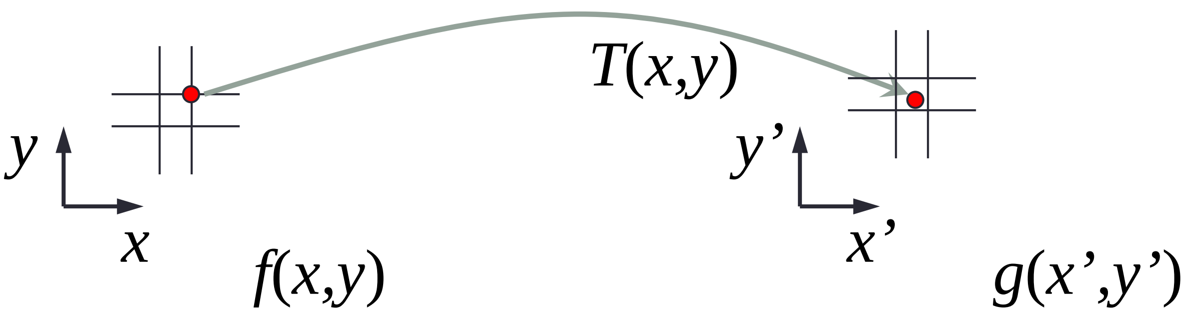


- Send each pixel $f(x,y)$ to its corresponding location
- $(x',y') = T(x,y)$ in the second image

Q: what if pixel lands "between" two pixels?

Slide from Alyosha Efros, CMU

Forward warping



- Send each pixel $f(x,y)$ to its corresponding location
- $(x',y') = T(x,y)$ in the second image

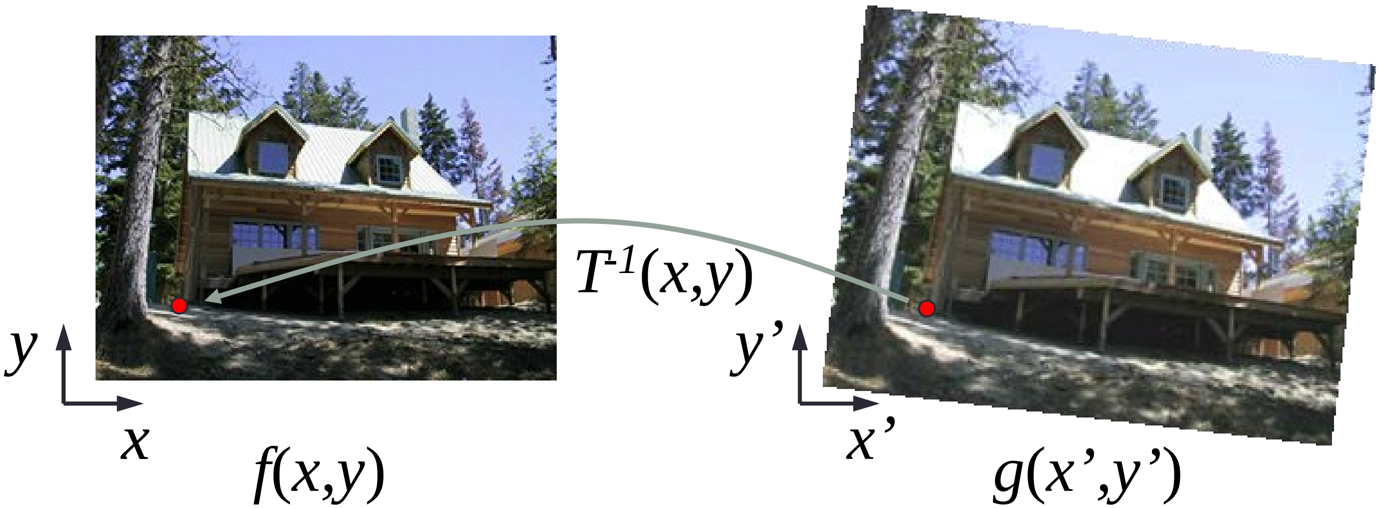
Q: what if pixel lands "between" two pixels?

A: distribute color among neighboring pixels (x',y')

- Known as "splatting"

Slide from Alyosha Efros, CMU

Inverse warping



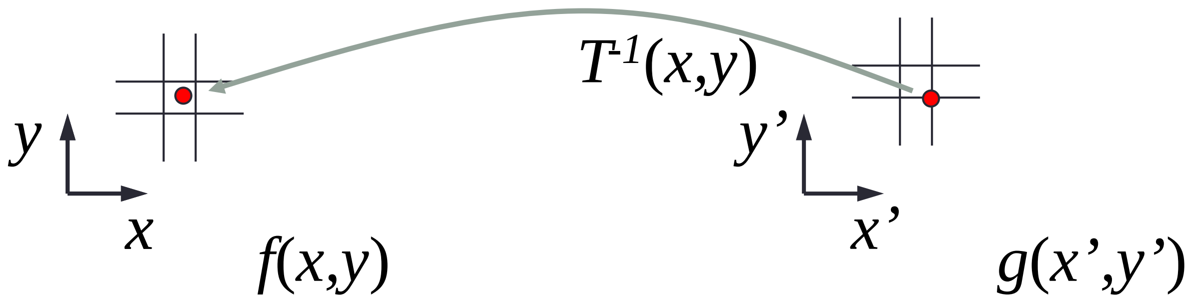
Get each pixel $g(x',y')$ from its corresponding location

$(x,y) = T^{-1}(x',y')$ in the first image

Q: what if pixel comes from "between" two pixels?

Slide from Alyosha Efros, CMU

Inverse warping



Get each pixel $g(x',y')$ from its corresponding location $(x,y) = T^{-1}(x',y')$ in the first image

Q: what if pixel comes from "between" two pixels?

A: *Interpolate* color value from neighbors

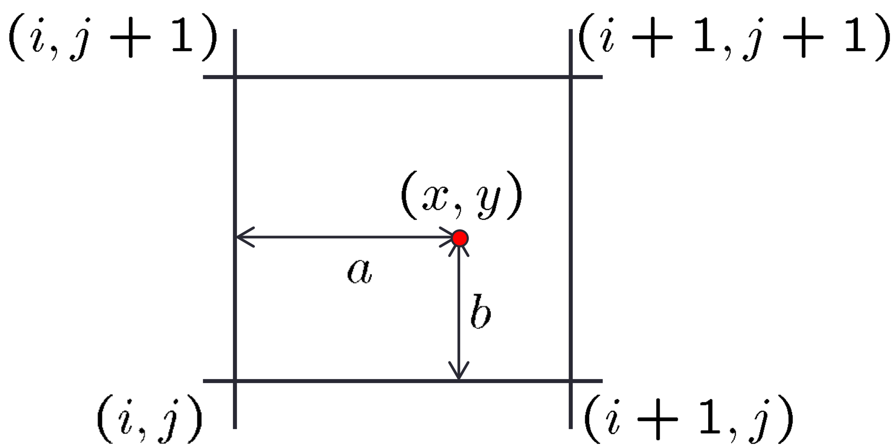
– nearest neighbor, bilinear…

Slide from Alyosha Efros, CMU

>> help interp2

Bilinear interpolation

Sampling at *f(x,y):*



$$
\begin{aligned}
f(x,y) = \; & (1-a)(1-b) && f[i,j] \\
& +a(1-b) && f[i+1,j] \\
& +ab && f[i+1,j+1] \\
& +(1-a)b && f[i,j+1]
\end{aligned}
$$

Slide from Alyosha Efros, CMU

Recap: How to stitch together a panorama (a.k.a. mosaic)?

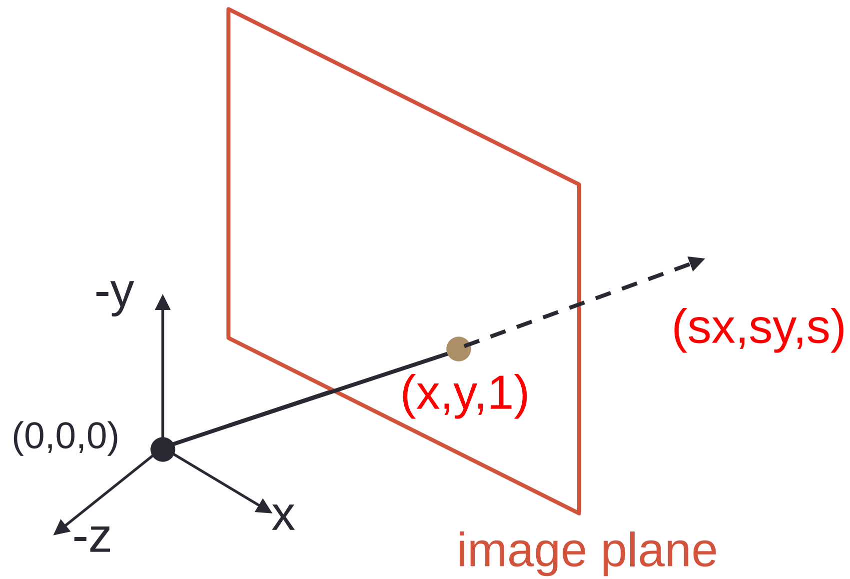
- Basic Procedure
  - Take a sequence of images from the same position
    - Rotate the camera about its optical center
  - Compute transformation (homography) between second image and first using corresponding points.
  - Transform the second image to overlap with the first.
  - Blend the two together to create a mosaic.
  - (If there are more images, repeat)

Source: Steve Seitz

Why projective geometry?

The projective plane

- What is the geometric intuition of using homogenous coordinates ?
  - a point in the image is a *ray* in projective space



- Each *point* (x,y) on the plane is represented by a *ray* (sx,sy,s)
  – all points on the ray are equivalent: (x, y, 1) ≅ (sx, sy, s)

Homogeneous coordinates

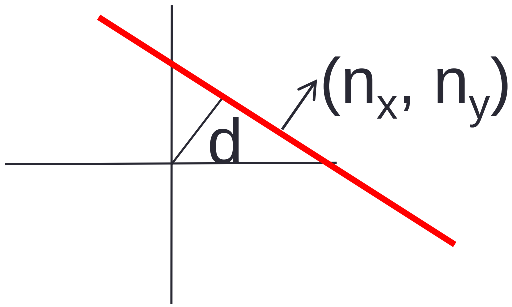
2D Points:

$$p = \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow p' = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad p' = \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \longrightarrow p = \begin{bmatrix} x'/w' \\ y'/w' \end{bmatrix}$$

2D Lines: $ax + by + c = 0$

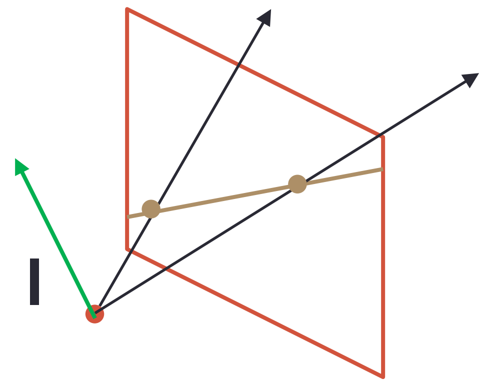
$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

$$l = \begin{bmatrix} a & b & c \end{bmatrix} \Rightarrow \begin{bmatrix} n_x & n_y & d \end{bmatrix}$$



Projective lines

- What does a line in the image correspond to in projective space?



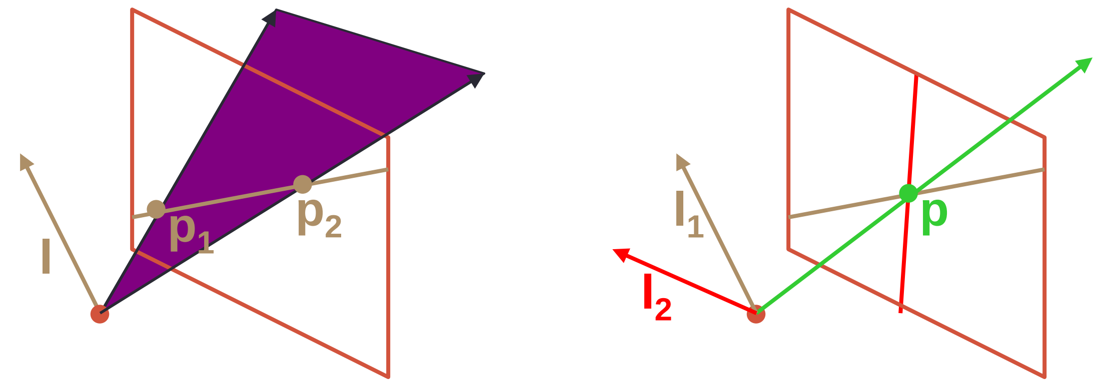
- A line is a *plane* of rays through origin
  – all rays (x,y,z) satisfying: ax + by + cz = 0

$$\text{in vector notation}: \quad 0 = \underbrace{\begin{bmatrix} a & b & c \end{bmatrix}}_{\mathbf{l}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\mathbf{p}}$$

- A line is also represented as a homogeneous 3-vector **l**

Point and line duality

- A line $\mathbf{l}$ is a homogeneous 3-vector
- It is $\perp$ to every point (ray) $\mathbf{p}$ on the line: $\mathbf{l}^T\mathbf{p}=0$



What is the line $\mathbf{l}$ spanned by rays $\mathbf{p}_1$ and $\mathbf{p}_2$?

- $\mathbf{l}$ is $\perp$ to $\mathbf{p}_1$ and $\mathbf{p}_2$ $\Rightarrow$ $\mathbf{l} = \mathbf{p}_1 \times \mathbf{p}_2$
- $\mathbf{l}$ is the plane normal

What is the intersection of two lines $\mathbf{l}_1$ and $\mathbf{l}_2$?

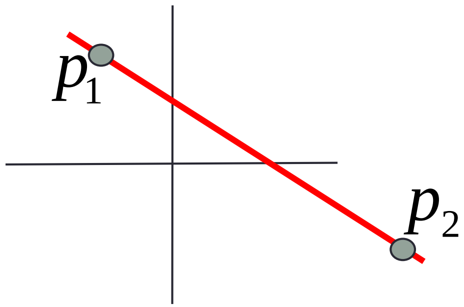
- $\mathbf{p}$ is $\perp$ to $\mathbf{l}_1$ and $\mathbf{l}_2$ $\Rightarrow$ $\mathbf{p} = \mathbf{l}_1 \times \mathbf{l}_2$

Points and lines are *dual* in projective space

- given any formula, can switch the meanings of points and lines to get another formula

Homogeneous coordinates

Line joining two points:

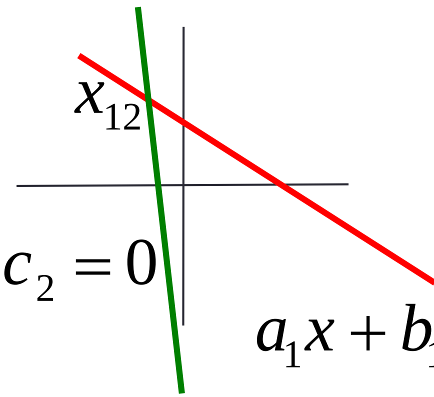


$$ax + by + c = 0$$

$$\left.\begin{array}{l} p_1 = \begin{bmatrix} x_1 & y_1 & 1 \end{bmatrix} \\ \\ p_2 = \begin{bmatrix} x_2 & y_2 & 1 \end{bmatrix} \end{array}\right\} \quad l = p_1 \times p_2$$

Homogeneous coordinates

Intersection between two lines:



$$a_2 x + b_2 y + c_2 = 0$$

$$a_1 x + b_1 y + c_1 = 0$$

$$\left.\begin{array}{l} l_1 = [a_1 \quad b_1 \quad c_1] \\ \\ l_2 = [a_2 \quad b_2 \quad c_2] \end{array}\right\} \; x_{12} = l_1 \times l_2$$

3D projective geometry

- These concepts generalize naturally to 3D
- Recall the equation of a plane:

$$aX + bY + cZ + d = 0$$

- Homogeneous coordinates
  - Projective 3D points have four coords: $\mathbf{p} = (wX, wY, wZ, w)$
- Duality
  - A plane $\mathbf{N}$ is also represented by a 4-vector $\mathbf{N} = (a,b,c,d)$
  - Points and planes are dual in 3D: $\mathbf{N}^T\mathbf{p} = \mathbf{0}$
- Projective transformations
  - Represented by 4x4 matrices T: P' = TP

3D to 2D: "perspective" projection

- Matrix Projection: $$\mathbf{p} = \begin{bmatrix} sx \\ sy \\ s \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \mathbf{MP}$$

You've already seen this.

What is *not* preserved under perspective projection?

What IS preserved?

What's next…

- Today – more projective geometry, the duality between points and lines in projective space

- Tuesday– using the projective geometry revisit 2-views:
  - Essential and Fundamental matrices